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Charge-changing transitions in an extended Lipkin-type model

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Abstract. Charge-changing transitions are considered in an extended Lipkin-Meshkov-Glick (LMG) model taking into account explicitly the proton and neutron degrees of freedom. The proton and neutron Hamiltonians are taken to be of the LMG form and, in addition, a residual proton-neutron interaction is included. Model charge-changing operators and their action on eigenfunctions of the model Hamiltonian are defined. Transition amplitudes of these operators are calculated using exact eigenfunctions and then the RPA approximation. The best agreement between the two kinds of calculation is obtained when the correlated RPA ground state, instead of the uncorrelated HF ground state, is employed and when the proton-neutron residual interaction, besides the proton-proton and neutron-neutron residual interactions, is taken into account in the model Hamiltonian.

PACS. 21.60.Fw Models based on group theory – 21.60.Jz Hartree-Fock and random-phase approximations – 13.75.Cs Nucleon-nucleon interactions

1 Introduction

The LMG [1] model was often used to test the validity of various microscopic approximation methods aiming to treat many-nucleon systems. The Hamiltonian of this model refers to nucleons of the same kind and contains a simple but still non-trivial two-body interaction between nucleons such that the model is solvable exactly within the quasi-spin $SU(2)$ algebra. In certain processes like, for instance, beta and double-beta decay, the proton-neutron (p-n) correlations in nuclei play, besides the like-nucleon (i.e. proton-proton (p-p) and neutron-neutron (n-n)) correlations, an important role as well. In the past many papers devoted to calculations concerning such processes have used RPA-type approaches in which all kinds of residual interactions between nucleons have been taken into account [2]–[9]. However, the competition between like- and unlike-nucleon residual interactions is not still well understood. Thus, it appears useful to have exactly solvable models distinguishing betwen protons and neutrons suitable to test RPA-type methods used in realistic calculations dealing with p-p, n-n and p-n correlations all together.

Recently, the LMG model was extended to take into account explicitly the protons and neutrons degrees of freedom [10]. The proton and neutron Hamiltonians were taken to be of the LMG form and, in addition, a residual p-n interaction was included. The exact and RPA spectrum were studied and the main conclusion was that the RPA method gives results closer to the exact ones when the p-n interaction besides the p-p and n-n interactions are all present in the model Hamiltonian.

The purpose of this paper is to show how charge-changing transitions simulating a nuclear $\beta^{\pm}$ can be studied in the framework of that extended p-n LMG model [10]. We defined transition operators that convert a neutron into a proton and viceversa, calculated their transition amplitudes by using firstly exact wave functions and then the RPA ones, and compared the emerging results.

2 The model Hamiltonian

In this section we give a brief description of our model, more details being given in [10]. The model consists in an N-nucleon system composed of two subsystems: one containing $N_{\rm p}$ protons, the other $N_{\rm n}$ neutrons. Inside each subsystem the nucleons of the same kind are distributed into two levels, each having an $N_{\rm p}(N_{\rm n})$-fold degeneracy, and separated by an energy $\epsilon_{\rm p}(\epsilon_{\rm n})$, respectively. Furthermore, the states in each subsystem are characterized by two quantum numbers: σ distinguishing between the lower ($\sigma = -1$) and upper ($\sigma = 1$) level and p (n) denoting all the other quantum numbers characterizing the proton (neutron) states of the level. For each p (n) there are two possible states, one in the lower level and the other in the upper level. The model Hamiltonian of the system can be expressed in terms of the quasi-spin operators for protons and neutrons:

$$H = \epsilon_{\rm p} J_z^{\rm p} + \epsilon_{\rm n} J_z^{\rm n} + \frac{1}{2} V_{\rm p} (J_+^{\rm p} J_+^{\rm p} + J_-^{\rm p} J_-^{\rm p}) + \frac{1}{2} V_{\rm n} (J_+^{\rm n} J_+^{\rm n} + J_-^{\rm n} J_-^{\rm n}) + V_{\rm pn} (J_+^{\rm p} J_+^{\rm n} + J_-^{\rm p} J_-^{\rm n}) \,, \quad (2.1)$$

where

$$J_+^\tau = \sum_\tau a_{\tau+}^\dagger a_{\tau-} \;; \quad J_-^\tau = \sum_\tau a_{\tau-}^\dagger a_{\tau+} \;;$$
$$J_z^\tau = \frac{1}{2} \sum_{\tau\sigma} \sigma a_{\tau\sigma}^\dagger a_{\tau\sigma}; \quad (\tau = p, n), \quad (2.2)$$

which satisfy the angular-momentum commutation relations. In the expressions of the $J_\pm$ operators the symbols $+$ and $-$ denote the two possible values $(+1, -1)$ of σ.

The terms proportional to $V_{\rm p}$ ($V_{\rm n}$) parameters scatter a pair of protons (neutrons) in the same level to the other level. In addition to these residual like-particle interactions which appear in the LMG model, we also consider terms proportional to $V_{\rm pn}$ which account for a p-n residual interaction. The interpretation of these terms is similar to the corresponding $V_{\rm p}$ and $V_{\rm n}$ terms described above but now the scattered pair is composed of one proton and one neutron, which can be excited within the corresponding subsystems.

The Hamiltonian matrix can be diagonalized exactly in the $SU(2) \otimes SU(2)$ basis

$$\Phi = |J_{\rm p} M_{\rm p}\rangle \otimes |J_{\rm n} M_{\rm n}\rangle \,, \quad (2.3)$$

where $J_{\rm p}$, $J_{\rm n}$ and $M_{\rm p}$, $M_{\rm n}$ are the total angular momenta and their projections onto the z-axis for the p and n subsystems, respectively. Since each nucleon has an $1/2$ angular momentum, in the no-interaction limit and assuming that in the ground state (g.s.) all possible p and n states of the lower level are occupied by one nucleon, the unperturbed (HF) g.s. reads:

$$|{\rm HF}\rangle = |\frac{N_{\rm p}}{2}, -\frac{N_{\rm p}}{2}\rangle |\frac{N_{\rm n}}{2}, -\frac{N_{\rm n}}{2}\rangle \,. \quad (2.4)$$

The first excited state of the system has one excited particle and is $N = (N_{\rm p} + N_{\rm n})$-fold degenerate. These states all have $J_z = -\frac{1}{2}N + 1$ and one can write formally:

$$|1^{st}\rangle = \frac{(J_+^{\rm p} + J_+^{\rm n})}{\sqrt{N}} |\frac{N_{\rm p}}{2}, -\frac{N_{\rm p}}{2}\rangle |\frac{N_{\rm n}}{2}, -\frac{N_{\rm n}}{2}\rangle = \frac{1}{\sqrt{N}} \Big(\sqrt{N_{\rm p}} |\frac{N_{\rm p}}{2}, -\frac{N_{\rm p}}{2} + 1\rangle |\frac{N_{\rm n}}{2}, -\frac{N_{\rm n}}{2}\rangle + \sqrt{N_{\rm n}} |\frac{N_{\rm p}}{2}, -\frac{N_{\rm p}}{2}\rangle |\frac{N_{\rm n}}{2}, -\frac{N_{\rm n}}{2} + 1\rangle \Big) \,, \quad (2.5)$$

where the norm factor in (2.5) was chosen such that $|1^{st}\rangle$ be normalized. The energy spectrum of (1) can be obtained exactly by diagonalizing the Hamiltonian matrix corresponding to the multiplet with ($J_{\rm p} = \frac{1}{2}N_{\rm p}$, $J_{\rm n} = \frac{1}{2}N_{\rm n}$) which contains the g.s. and all the states of interest. After diagonalization the exact wave functions of the system will be linear combinations of wave functions of the basis (2.3).

Now we treat our model Hamiltonian (1) within the RPA method. The RPA phonon operator can be defined as follows:

$$\Gamma^\dagger = \frac{X\Theta^+ - Y\Theta^-}{\sqrt{\langle {\rm HF}|[\Theta^-, \Theta^+]|{\rm HF}\rangle}},$$
$$\Theta^+ = J_+^{\rm p} + J_+^{\rm n} \;\; ; \quad \Theta^- = J_-^{\rm p} + J_-^{\rm n} \quad (2.6)$$

X and Y are the RPA amplitudes. The norm appearing in the expression of Γ^+ is taken such that the first excited states be normalized to unity and has the value $\sqrt{N}$. The correlated RPA g.s. for our system, obtained by solving the equation $\Gamma|{\rm RPA}\rangle = 0$ [11], and the first (normalized) excited state reads:

$$|{\rm RPA}\rangle = N_0 {\rm exp} \left[-\frac{1}{2N} \sqrt{\left(\frac{\epsilon - \Omega}{\epsilon + \Omega} \right)} \Theta^+ \Theta^+ \right] |{\rm HF}\rangle \quad (2.7)$$

$$|1^{st}\rangle = \Gamma^\dagger |{\rm RPA}\rangle \quad (2.8)$$

where

$$N_0 = \left(1 + \frac{N-1}{2N} \frac{\epsilon - \Omega}{\epsilon - \Omega} \right) \quad (2.9)$$

is the normalization factor.

3 Charge-changing transitions

In the framework of our model we consider now transitions in which a neutron transforms into a proton (or viceversa), such that the total number of nucleons remains constant. Such transitions can be associated to nuclear beta decay.

Let us define the following transition operators:

$$\hat{M}^+ = \chi^+ \sum_{p,n,\sigma,\sigma'} a_{p\sigma}^\dagger a_{n\sigma'};$$
$$\hat{M}^- = \chi^- \sum_{p,n,\sigma,\sigma'} a_{n\sigma'}^\dagger a_{p\sigma} = (\hat{M}^+)^+ \quad (3.1)$$

The $\hat{M}^+$ operator can destroy one neutron from any of the states of the two levels of the neutron subsystem and create one proton into any of the states of the two levels of the proton subsystem. Such an operator can connect states belonging to systems with the same total number of nucleons, N, but differing from each other in the number of protons and neutrons in their respective subsystems. For example, it can connect states of the system $|N_{\rm p}\rangle|N_{\rm n}\rangle$ with states belonging to the system $|N_{\rm p}+1\rangle|N_{\rm n}-1\rangle$. This transition simulates a nuclear β^- decay. The factor χ^+ in front can be interpreted as the strength of the transition. We imagine that the creation of the proton can occur only in a p state which is unoccupied (free), both in the lower and in the upper level of the p subsystem. Hence, we suppose that such free states exist in the lower level (with an accompanying free upper level) even in the g. s. of the system. The existence of free states in the subsystems of

the N-nucleon system does not affect the consistency of the model. Moreover, their presence does not influence either the possible values of the total angular momentum (which only depend of the number of particles) or the Hamiltonian eigenvalues. The only quantity which is affected is the degeneracy of the g. s. and excited states of the system which now increases, but this effect can easily be accounted for. Allowing this degree of freedom it will produce also a change in the values of the normalization factors of (2.4) and (2.5), which have to be re-calculated in order to keep the corresponding states normalized to unity. As an intuitive physical correspondence, one can imagine to associate the p (or n) subsystem with a p (or n) closed or open nuclear shell. The $\hat{M}^-$ operates in a similar manner as $\hat{M}^+$, but now one proton is transformed into one neutron and the transition simulates a nuclear β^+ decay.

We proceed further to the calculation of the g.s. to g.s. and g.s. to $|1^{st}\rangle$ transition amplitudes. Consider an N-nucleon system having $N_{\rm p}$ protons and $N_{\rm n}$ neutrons in its two subsystems. Assume the particular case when in the proton subsystem we have $N_{\rm p}+1$ possible states, while in the neutron subsystem the number of possible n states is equal to the number of neutrons, $N_{\rm n}$. Generalization to the case of more than one free (hole) states in any of the subsystems can be easily made, but for simplicity we restrict ourselves to this case. The operator $\hat{M}^+$ can connect states of this system only with states of a system with the same N but with $N_{\rm p}+1$, $N_{\rm n}-1$. The simplest of these systems is the one which has in the g. s. all the possible p states occupied (in the p subsystem), while, in the n subsystem there is one (free) hole. In this particular case it is easy to see that the normalization factors for the g.s. are:

$$|N_{\rm p}({\rm 1h}); N_{\rm n}\rangle = \frac{1}{\sqrt{N_{\rm p}+1}}|\frac{N_{\rm p}}{2};-\frac{N_{\rm p}}{2}\rangle|\frac{N_{\rm n}}{2};-\frac{N_{\rm n}}{2}\rangle \quad (3.2)$$

$$|N_{\rm p}+1; N_{\rm n}-1({\rm 1h})\rangle = \frac{1}{\sqrt{N_{\rm n}}}|\frac{N_{\rm p}+1}{2};-\frac{N_{\rm p}+1}{2}\rangle |\frac{N_{\rm n}-1}{2};-\frac{N_{\rm n}-1}{2}\rangle \quad (3.3)$$

The symbol (1h) appearing in the above formulae denotes the hole associated to the free state in the corresponding subsystem. The first excited states of these systems will have the normalization factors of (3.2) and (3.3) multiplying the factor $\sqrt{N}$ of (2.5).

The $\hat{M}^-$ operator can connect states of the N-nucleon system having $N_{\rm p}$ protons and $N_{\rm n}$ neutrons with the N-nucleon system having $N_{\rm p}-1$ protons and $N_{\rm n}+1$ neutrons. Assuming again the existence of only one hole in the corresponding subsystems, we get similar normalization factors as in the case of $\hat{M}^+$, but with p changed to n.

Next, we calculate charge-changing transitions between the ground states and between the g. s. and the first excited state. First we performed the diagonalization of the Hamiltonian and got the exact wave functions as linear combination of the unperturbed states in the full $SU(2)$ basis (2.3). Then, in the calculation of the transition amplitudes we used the following approximation: we considered our exact g.s and first excited wave functions be linear combinations of only unperturbed states having at most 2p-2h excitations. In the RPA framework this corresponds to the use of the correlated $|{\rm RPA}\rangle$ g.s. as the g.s of the system. It is worth mentioning that this approximation is well justified by the very small values of the mixing coefficients of the exact g.s. and first excited state with unperturbed states having more than 2p-2h excitations. From a numerical estimation we found that the exact calculation of the transition amplitudes obtained with this approximation differs within 2% from the exact result when all possible excitations are included. The reason of using this approximation was to save much labour. Denoting the g.s. as Ψ_0 and the first excited state as Ψ_1 one obtains the following results

a) Exact results

$$T_{00} = \langle\Psi_0'|\hat{M}^+|\Psi_0\rangle = \sqrt{N_{\rm n}(N_{\rm p}+1)}c_{11}c_{11}'\chi^+\,, \quad (3.4)$$

$$T_{01} = \langle\Psi_1'|\hat{M}^+|\Psi_0\rangle = \left(\sqrt{N_{\rm n}}c_{11}c_{21}' + \sqrt{N_{\rm p}}c_{22}c_{21}' + \sqrt{2(N_{\rm p}+1)}c_{13}c_{12}'\right)\chi^+\,. \quad (3.5)$$

b) RPA results

$$T_{00} = \langle{\rm RPA}'|\hat{M}^+|{\rm RPA}\rangle = \sqrt{N_{\rm n}(N_{\rm p}+1)}\chi^+ \times\left[\left(1+\frac{N-1}{2N}\frac{\epsilon-\Omega}{\epsilon+\Omega}\right)\left(1+\frac{N-1}{2N}\frac{\epsilon-\Omega'}{\epsilon+\Omega'}\right)\right]^{-1/2}, \quad (3.6)$$

$$T_{01} = \langle 1^{st}|\hat{M}^+|{\rm RPA}\rangle = \sqrt{\frac{N_{\rm n}(N_{\rm p}+1)}{N}}\left(1+\frac{N-1}{2N}\frac{\epsilon-\Omega}{\epsilon+\Omega}\right)^{-1/2} \times\left(1+\frac{N-1}{2N}\frac{\epsilon-\Omega'}{\epsilon+\Omega'}\right)^{-1/2}\left(1-\frac{N-1}{N}\sqrt{\frac{\epsilon-\Omega}{\epsilon+\Omega}}\right) \times\left(X'+\frac{N-1}{N}\sqrt{\frac{\epsilon-\Omega'}{\epsilon+\Omega'}}Y'\right)\chi^+\,. \quad (3.7)$$

where $|RPA\rangle$ and $|RPA'\rangle$ represent the correlated RPA g.s. functions. Above the states denoted by prime refer to the final system having $(N_{\rm p}+1; N_{\rm n}-1)$, while the states without any additional index refer to the initial system having $(N_{\rm p}; N_{\rm n})$ nucleons. The coefficients c_{ij} are related to the decomposition of the g.s. and first excited state wave functions in terms of the unperturbed wave functions of the basis (2.3). One notes that in the formulae (3.4-3.7) enter only those coefficients which are connected to the unperturbed wave functions containing at most 2p-2h excitations.

In Figs. 1, 2 we display the transitions between the ground states and between the g. s. and the first excited

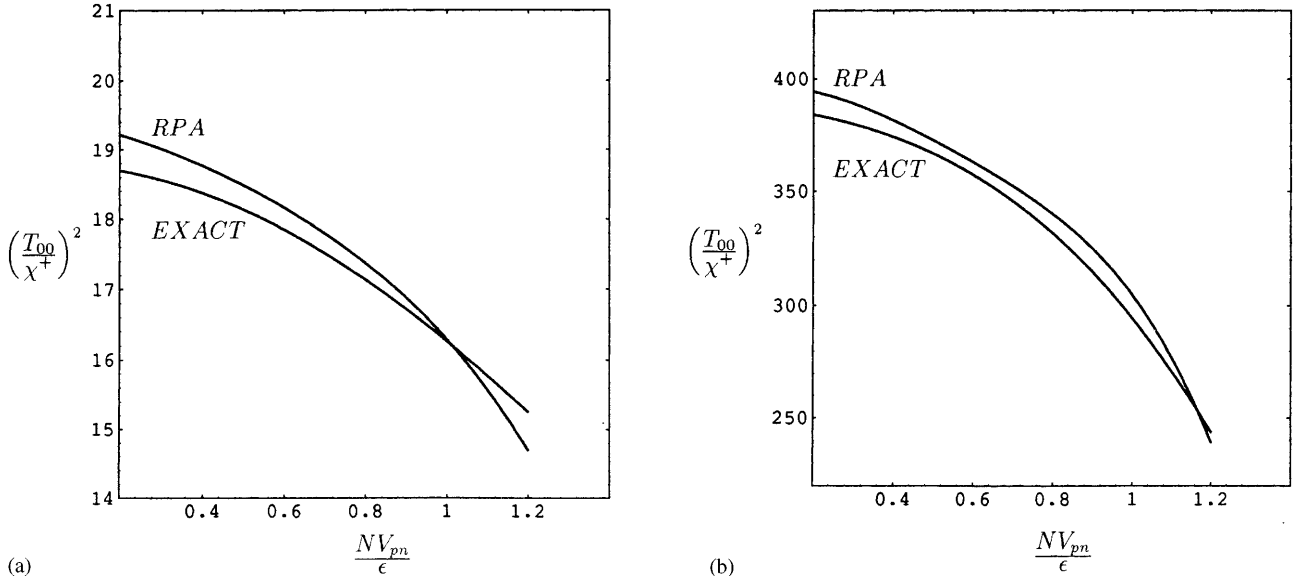

Fig. 1. Square of the charge-changing transition amplitudes between ground states as a function of the interaction parameter $\frac{NV_{\rm pn}}{\epsilon}$ at $\frac{NV}{\epsilon} = 0.8$ for (**a**) transitions from $(N_{\rm p} = 4; N_{\rm n} = 4)$ to $(N_{\rm p} = 5; N_{\rm n} = 3)$, (**b**) transitions from $(N_{\rm p} = 20; N_{\rm n} = 20)$ to $(N_{\rm p} = 21; N_{\rm n} = 19)$

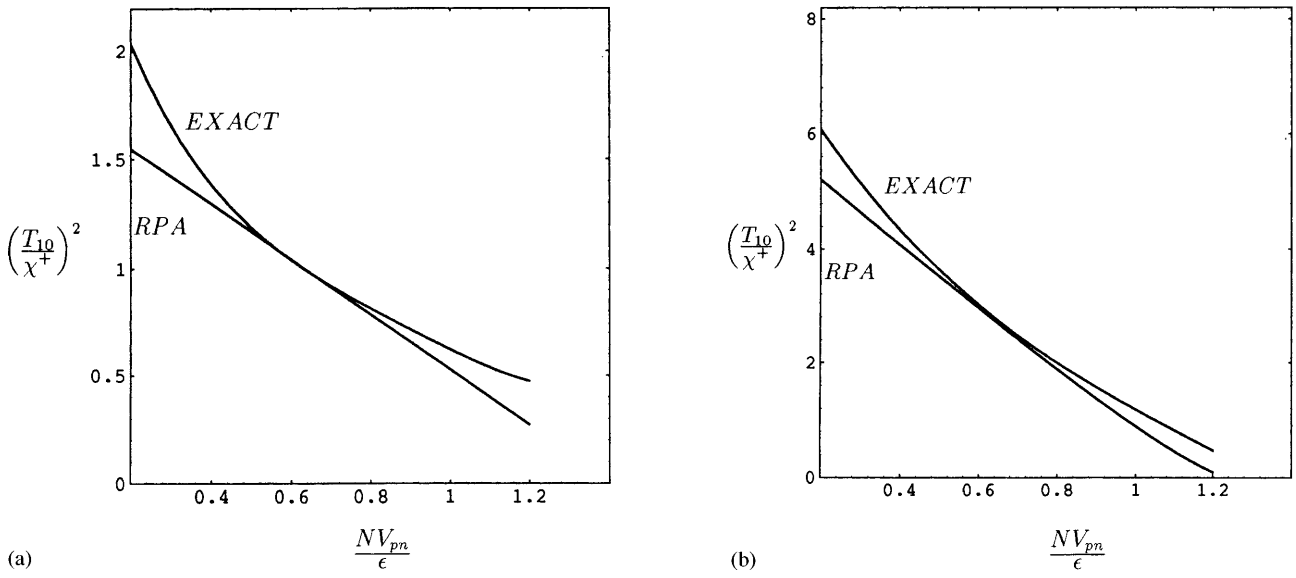

Fig. 2. Square of the charge-changing transition amplitude between the ground state and the first excited state as a function of the interaction parameter $\frac{NV_{\rm pn}}{\epsilon}$ at $\frac{NV}{\epsilon} = 0.8$ for (**a**) transitions from $(N_{\rm p} = 4; N_{\rm n} = 4)$ to $(N_{\rm p} = 5; N_{\rm n} = 3)$, (**b**) transitions from $(N_{\rm p} = 20; N_{\rm n} = 20)$ to $(N_{\rm p} = 21; N_{\rm n} = 19)$

state as function of the model parameter $NV_{\rm pn}/\epsilon$ and for a fixed value of the NV/ϵ, in the case of the systems with $N = 8$ and $N = 40$, as it is indicated on each figure. Figures 1a, b display the g.s to g.s. β^--type transitions. Firstly, one remarks the good agreement between the exact and RPA results. The agreement seems to improve in the case of a system with a larger number of nucleons (N=40), in accordance with the fact that the RPA works better for systems with a large number of nucleons, where collective effects manifest stronger. Furthermore, one observes that at a certain value of the model parameters $NV_{\rm pn}/\epsilon$ and NV/ϵ, different from zero, the exact and the RPA calculation for these transitions give the same results. This shows the presence of both kinds (i.e. like and unlike- particle) of residual interactions further improves the RPA results. The same conclusions also emerge by analysing the Figs. 2a, b where we plotted transitions from the g.s. to the first excited state. In this case, the agreement between exact and RPA results is clearer better in the case $N = 40$. Moreover, for these transitions there is a whole region of values of the model parameters (different from zero) where the exact and RPA results are very close

to each other. Thus, in this case, the presence of both kinds of residual interactions is also important in getting exact and RPA results very close to each-other.

It is worth mentioning that the features presented above are generally the same for other values of the model parameters (i.e. the total number of nucleons and the NV/ϵ parameter). Also, here we limited our discussion to the transitions induced by the M^+ simulating a β^- nuclear transition. Transitions induced by the M^- operator, simulating a nuclear β^+ transition as well as transitions between other initial and final states can also be calculated in a similar manner.

4 Conclusions

Concluding, charge-changing transitions were studied in the framework of an extended LMG model taking into account explicitly the proton and neutron degrees of freedom. Model charge-changing transition operators simulating nuclear β decay and their action on eigenfunctions of the model Hamiltonian were defined. The spectrum of the model Hamiltonian was obtained by diagonalization in the full $SU(2) \otimes SU(2)$ basis (2.3). However, the transition amplitudes of these operators were calculated by limiting our exact basis to be a superposition of unperturbed states having at most 2p-2h excitations and, accordingly, we considered the correlated RPA g.s.. We found the RPA results for charge-changing transition amplitudes get close to the exact ones in the case of a system with more nucleons when the collective effects manifest stronger. Moreover, the presence of the p-n residual interaction besides the p-p and n-n residual interactions in the model Hamiltonian further improves the agreement between the exact and RPA calculation. All these features emerging from the study of charge-changing transitions in the framework of an exactly solvable model encourage the employment, in realistic situations, of RPA-type approaches, where like- and unlike-nucleon residual interactions are taken into account on the same footing.

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